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Benchmark Problem Solution by Quantitative Feedback Theory

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Quantitative feedback theory is used to synthesize compensators satisfying various design specifications stipulated in the "benchmark problem" statement. The primary goal in this design technique is to keep the cost of feedback, as measured by the controller bandwidth, as low as possible while realizing the specified performance specifications in the presence of plant uncertainty (parametric in the low frequencies and unstructured in the high frequencies) and external disturbances. The order of the compensators is directly related to the performance objectives and bandwidth limitations. A fifth-order compensator for design 1 and a ninth-order compensator for design 2 were obtained for satisfying the performance specifications with finite controller bandwidth. If sensor noise is not considered, then these compensator orders can be reduced.

I. Introduction

IN this paper, we employ quantitative feedback theory (QFT) to synthesize compensators, satisfying various design specifications stipulated in the benchmark problem[1] detailed in Sec. III. Minimizing the cost of feedback, as measured by the amount of controller bandwidth, is the main objective of QFT. It emphasizes the fact that feedback is only necessary because of uncertainty and that the amount of feedback should therefore be directly related to the extent of plant uncertainty and unknown external disturbances. Consequently, robustness has been a major consideration in QFT since its introduction by Isaac Horowitz in the early sixties. In modern control literature, however, serious robustness considerations did not materialize until the late seventies, despite the fact that feedback is only necessitated by plant uncertainties and external disturbances.

QFT is a unified theory that emphasizes the use of feedback for achieving desired system performance tolerances despite plant uncertainty and plant disturbances. QFT *quantitatively* formulates these two factors in the form of 1) sets $\boldsymbol{T_R} = \{ T_R \}$ of acceptable command or tracking input-output relations and $\boldsymbol{T_D} = \{ T_D \}$ of acceptable disturbance input-output relations, and 2) a set $\boldsymbol{P} = \{ P \}$ of possible plants. The control objective is to guarantee that the control ratio $T_R = Y/R$ is a member of $\boldsymbol{T_R}$ and $T_D = Y/D$ is a member of $\boldsymbol{T_D}$, for all P in $\boldsymbol{P}$, where Y is the output, R is the reference input, and D is the output disturbance shown in Fig. 1.

The present benchmark problem can be solved quite easily using basic ideas in QFT. In light of the required performance specifications, the QFT problem becomes a simple robust stabilization subject to a certain phase margin constraint.

This paper is organized as follows. In Sec. II, the classical QFT problem is stated and the methodology is briefly outlined. The solutions to the benchmark problem are developed in Sec. III followed by a discussion in Sec. IV and conclusions in Sec. V.

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II. Quantitative Feedback Theory

QFT Problem

The QFT problem can be stated as follows: There is given an uncertain family of finite dimensional linear time invariant plants:

$$P = P_L(\alpha,s) \cup P_H(s)$$

where $P_L(\alpha,s)$ represents the low- to mid-frequency plant-set with parametric uncertainty $\alpha \in \Omega$, where Ω is a compact parameter space and $\alpha \in \boldsymbol{R}^p$ is a p vector of uncertain parameters. The $P_H(s)$ represents the high-frequency uncertain plant-set model to account for such unknowns as unmodeled dynamics, measurement errors, and unknown parameter variations. If the relative degree of $P_L(\alpha,s)$ is r, $\forall \alpha \in \Omega$, it is asumed that $P_H(s)$ satisfies

$$|P_H(j\omega)| \leq \frac{k_L(\alpha)}{\omega^r}, \qquad \forall \omega \geq \omega_h$$

where 1)

$$k_L(\alpha) = \lim_{|s| \to \infty} s^r P_L(\alpha,s)$$

where $k_L(\alpha)$ is called the plant high-frequency gain set and 2) ω_h is an effective high frequency from which $P_L(\alpha,s)$ is structurally unconstrained and effectively allowed to roll off at an arbitrary rate. It is from this frequency upward that P is assumed to have arbitrary phase uncertainty.

There is also given an ideal target closed-loop transfer function $T_o(s)$ and an ideal disturbance target transfer function

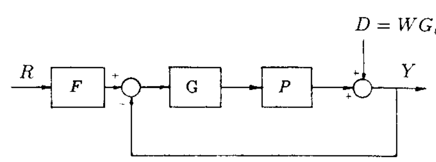

Fig. 1 Two degree-of-freedom control configuration.

$T_D^o(s)$. The high-frequency plant dynamics $P_H(s)$ places a stability margin constraint on the controlled system. Let

$$T(\alpha,s) = H(\alpha,s)F(s) \tag{1}$$

$$H(\alpha,s) = \frac{L(\alpha,s)}{1+L(\alpha,s)} \tag{2}$$

$$L(\alpha,s) = P_L(\alpha,s)G(s) \tag{3}$$

where H is the closed-loop transfer function, F is the prefilter, L is the loop transfer function, and G is the controller.

The QFT problem is to find an admissible pair of strictly proper rational and preferably stable transfer functions $[G(s),F(s)]$ in the two degree-of-freedom configuration (Fig. 1) such that the following conditions are satisfied with some measure of optimality:

1) $T(\alpha,s)$ is stable $\forall \alpha \in \Omega$
2) $\max_{\alpha\in\Omega}|T(\alpha,s)-T_o(s)| \le |\delta_T(s)|$, $\forall s$
3) $\max_{\alpha\in\Omega}|T_D(\alpha,s)| \le |T_D^o(s)| = |\delta_D(s)|$, $\forall s$

where $\delta_T(s)$ and $\delta_D(s)$ are prespecified. It is assumed that P satisfies the following conditions:

1) P is path connected. That P is connected implies that $[k_L(\alpha)/k_L(\alpha_o)] > 0 \forall \alpha_o$, $\alpha \in \Omega$.
2) Every P has the same number of nonminimum phase zeros and unstable poles.

Classical QFT

In classical QFT[2], the previous design problem is solved almost entirely graphically on the Nichols chart through the following procedure:

1) For $P_L(\alpha,j\omega)$ at any frequency $\omega \le \omega_h$, the mapping $P_L(\alpha,j\omega): \Omega \to R(g,\phi)$ generates a compact set $R(g,\phi)$ in the gain-phase plane (g,ϕ) called the plant *template* at that frequency.

2) By use of $R(g,\phi)$, the performance specifications $[\delta_T(\omega),\delta_D(\omega)]$ are mapped into a $B_p(\omega,\psi)$ that is a constraint on the nominal loop transfer function $L_o(j\omega,\psi) = L(\alpha_o, j\omega) = [P_L(\alpha_o)G](j\omega,\psi)$ at that frequency according to

$$B_p(\omega,\psi) = \Big\{\inf|L_o(j\omega,\psi)| \text{ s.t. } \max_{\alpha\in\Omega}|T - T_o|_{j\omega} \le \delta_T(\omega) \text{ and } \max_{\alpha\in\Omega}|T_D|_{j\omega} \le \delta_D(\omega)\Big\}$$

where ψ denotes an arbitrary phase angle for $L_o(j\omega)$ typically in the range [0, 360] deg since the actual phase of $L_o(j\omega)$ is as yet unknown.

3) The high-frequency bound on the unstructured plant $P_H(j\omega)$ maps the parameter space Ω into a stability boundary according to

$$P_H(j\omega): \Omega \to B_s(\omega), \qquad \text{for } \omega \ge \omega_h$$

When $\omega_h \to \infty$, $B_s(\omega) \to B_s(\infty)$, which is called the universal high-frequency boundary.

4) The stability specification is given as constraints $B(g_m,\phi_m)$ on the gain margin g_m and phase margin ϕ_m around the critical point on the Nichols chart.

5) The union of $B(g_m,\phi_m)$ and $B_s(\infty)$ given by

$$B_\infty(g_m,\phi_m) = B_s(\infty) \cup B(g_m,\phi_m)$$

generates a contour around the critical point (0 dB, −180 deg) called the universal high-frequency contour (U contour).

6) Finally, the design constraints on $L_o(j\omega)$ are given by $B_\infty(g_m,\phi_m) \cup \{\cup_{0\le\psi\le 360\,\text{deg}} B_p(\omega,\psi)\}$. Any $L_o(j\omega)$ that is QFT admissible must satisfy

$$|L_o(j\omega,\psi)| \ge B_p(\omega,\psi) \qquad \forall \omega \le \omega_h$$

and

$$|L_o(j\omega)| \ge B_\infty(g_m,\phi_m) \qquad \forall > \omega_h$$

Subject to these constraints, $L_o(j\omega)$ can be rolled off as rapidly as desired by the addition of extra poles after $\omega = \omega_h$.

An optimal loop transfer function L_{opt} must satisfy the constraints with the minimum possible gain at every frequency and phase (ψ).[3] Consequently, $L_{\text{opt}} \in \cup_{\psi\in[0,2\pi]} B_p(\omega,\psi)$ for all $\omega \le \omega_h$. This optimum (if it exists), in general, can be shown to be unique and nonrational. The QFT loop shaping reduces to obtaining finite order suboptimal loop transmission functions that approach L_{opt} as closely as desired. Once a suitable L_o is found, the prefilter can be easily determined.[4] (Since the solution to the present benchmark problem does not involve tracking, a prefilter is not needed.)

III. Benchmark Problem and Specifications

Consider the two-mass spring system shown in Fig. 2, which is a generic model of an uncertain dynamical system with noncollocated actuators and sensors. A control force acts on mass m_1, and the position of mass m_2 is measured, resulting in a noncollocated problem. This undamped mass-spring system has the following transfer functions relating the control force on the first mass to the position of the second mass, and the disturbance force $w(t)$ on the second mass to its position, respectively:

$$\frac{Z(s)}{U(s)} = \frac{k}{m_1 s^2\big\{m_2 s^2 + [1 + (m_2/m_1)]k\big\}} \tag{4}$$

$$\frac{Z(s)}{W(s)} = \frac{(m_2 s^2 + k)}{m_1 s^2\big\{m_2 s^2 + [1 + (m_2/m_1)]k\big\}} \tag{5}$$

where the masses m_1 and m_2, and the spring stiffness k can be unknown but bounded. Three separate but related controller design problems are to be investigated. The specifications for each of the designs of the benchmark problem are given next.[1]

Problem 1

Design a constant gain linear feedback compensator having the following properties:

1) The closed-loop system is stable for $m_1 = m_2 = 1$ and $0.5 < k < 2.0$.
2) For $w(t)$ = unit impulse at $t = 0$, the performance variable z has a settling time of about 15 s for the system $m_1 = m_2 = k = 1$.
3) The measurement noise $v(t)$ is to be characterized to reflect realism and practical control design.
4) Achieve reasonable performance/stability robustness.
5) Minimize controller effort.
6) Minimize controller complexity.

Problem 2

Problem 2 is the same as problem 1 except that in place of 1) the controller must be able to achieve asymptotic rejection of a sinusoidal disturbance of 0.5 rad/s with a 20-s settling time for $m_1 = m_2 = 1$ and $0.5 \le k \le 2$.

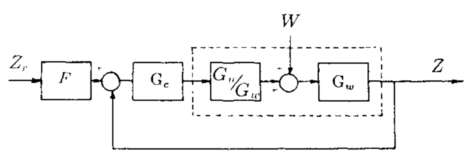

Fig. 2a Mass-spring system block diagram for control.

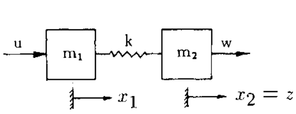

Fig. 2b Mass-spring system.

Problem 3

Problem 3 is the same as problem 1 except that a stability robustness measure with respect to the three uncertain parameters m_1, m_2, and k is maximized.

Remark

In executing the QFT design for this benchmark problem, the nominal parameter values were chosen as $m_1 = m_2 = 1$ and $k = 0.5$. This choice was deliberately made to demonstrate the efficacy of the technique in achieving performance specifications. It should be emphasized that in QFT the nominal plant can be defined arbitrarily by any set of specific parameter values within the allowed regions. Note that in some of the recent robust control techniques (H_∞, μ-synthesis) such a free choice of the nominal plant may not be possible.[5]

QFT Design for Problem 1

On examining the specifications it becomes clear that there are no hard time domain or frequency domain constraints stipulated for the system. Specifications call for about 15-s settling time for an impulse disturbance, a minimum control effort, and a reasonable characterization of sensor noise. Consequently, quantitative performance guarantees are not needed. Hence, it is not necessary to go through an elaborate QFT design. The qualitative specification on settling time will be incorporated by guaranteeing a minimum phase margin for the closed-loop system. Although there is no direct relationship between settling time and phase margin, as a rule of thumb we specify a phase margin to handle the settling time constraint. However, it must be emphasized that, although the specifications only call for about 15-s settling time for the nominal plant defined by $m_1 = m_2 = k = 1$, we will assure a minimum phase margin for the entire family of plants defined by Eqs. (4) and (5). Discussed next is how we settle on a minimum phase margin that will, in an approximate sense, translate into an appropriate settling time.

Minimum Phase Margin

Suppose a nominal loop transfer function $L_o(s)$ has been synthesized. With that nominal loop transfer function let the closed-loop system disturbance transfer function be

$$\frac{Z(s)}{W(s)} = \frac{G_w}{I + L_o(s)[P(s)/P_o(s)]} \tag{6}$$

Equation (6) represents an infinite family of impulse responses corresponding to every possible parameter combination of $P(s)$. Our objective in the QFT design is to guarantee that every such impulse response settles within 15 s, although the specifications only require the nominal plant ($m_1 = m_2 = k = 1$) to satisfy the settling time condition. To have such a realization, as a rule of thumb, we bound the entire family of disturbance transfer functions defined in Eq. (6) by a target transfer function that is known to have a settling time of 15 s. Let that target transfer function be $T_D^o(s)$. Then we use the basic QFT philosophy and bound the entire family of disturbance transfer functions by the target at every frequency up to the frequency where the open loop is below -10 dB or a decade beyond the range of validity of the plant model. The argument then goes something like this. If all of the plant transfer functions are bounded by the target at each frequency, then the corresponding impulse responses will also be bounded by the impulse response of the target provided the set of plant transfer functions are path connected. Of course the latter argument is not strictly correct. Nevertheless, with the given qualitative specifications, we are really not interested in bounding the impulse responses. All that we are interested in is assuring an approximate settling time for the entire class. Suppose the target transfer function is chosen as the following second-order system:

$$T_D^o(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7}$$

then its settling time t_s can be approximated by the formula

$$t_s \approx \frac{4}{\zeta\omega_n}$$

So, for a 15-s settling time we need

$$\zeta\omega_n \geq 0.27$$

Note that the last expression implies that, if the bandwidth of the system is large, then a relatively smaller damping ratio ζ can be tolerated. Since we are not specifically required to restrict ω_n, we arbitrarily selected $\omega_n = 0.9$. This was based on the fact that the nominal plant has a pair of undamped poles at $\pm\sqrt{2}j$, implying that it may not be possible to increase the system bandwidth beyond $\sqrt{2}$. So our goal in the QFT design is to make sure that the closed-loop system has an effective bandwidth of about 0.9, with a corresponding damping ratio of 0.30. A damping ratio of 0.30 is approximately equal to a phase margin of 30 deg. So the QFT objective can be stated as follows: *Determine a nominal loop transfer function that guarantees a minimum phase margin of 30 deg for the entire family of transfer functions.*

Remark

It can be seen that if the pointwise satisfaction of the constraint

$$\left|\frac{G_w(j\omega)}{1 + L_o[P(j\omega)/P_o(j\omega)]}\right| \leq |T_D^o(j\omega)|$$

at each frequency were to imply analogous bounding in the time domain, then we are done. Since it is not necessarily true, the phase margin constraint is added as a qualifier.

Template Generation

Consider the uncertain plant-set

$$P(s) = \frac{k}{s^2[s^2 + 2k]}$$

with $k \in [0.5, 2]$ for generating the templates needed to determine frequency domain bounds on the nominal loop transfer function amplitude $|L_o(\omega)|$ at every phase angle, for each frequncy ω. The templates will have either a phase angle of -180 deg or a phase angle of -360 deg, depending on whether the frequency is less than $\sqrt{2k}$ or greater than $\sqrt{2k}$. Shown in Fig. 3 (where x denotes the nominal plant $m_1 = m_2 = 1$, $k = 0.5$) are three templates corresponding to $\omega = 0.2$, 1.2, and 3 rad/s. The template for $\omega = 0.2$ rad/s consists only of points along the vertical through -180 deg. This is so because over the range of variation of k the smallest possible value of the undamped natural frequency is 1.0 rad/s. In fact, for any frequency below 1 rad/s the plant template is a vertical through -180 deg. Note that for $\omega \ll 1$, $P(j\omega) \to -0.5/\omega^2$, so the template is very close to being a single point. As ω increases, the template gets longer, being only 0.27 dB long at $\omega = 0.2$ rad/s, 1.16 dB at $\omega = 0.4$ rad/s, 30.5 dB at $\omega = 0.99$ rad/s, and approaching infinity as $\omega \to 1$. In this range of $\omega < 1$, the nominal is the highest point, marked x in Fig 3a.

For $\omega \in (1,2)$ the templates consist of two vertical lines 180 deg apart. The nominal is the lowest point on the left leg, marked x. As k increases from its nominal value of 0.5, $P(j\omega)$ moves upward to infinity at $k = \omega^2/2$, returns from infinity, through the right leg to its minimum value at $k = 2$. For example, the template corresponding to $\omega = 1.2$ rad/s consists of points along both verticals through -180 and -360 deg. For $k < 0.72$ the points lie along -360 deg, reaching infinity at $k = 0.72$ and returning through the vertical at -180 deg for $k > 0.72$. The points transfer from -180 to -360 deg through ∞.

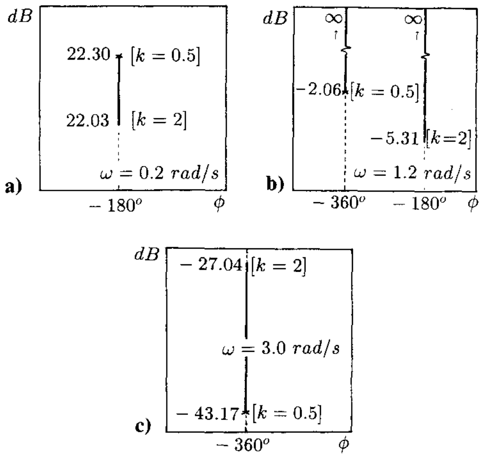

Fig. 3 Templates (x-nominal plant): a) $\omega=0.2$ rad/s, b) $\omega=1.2$ rad/s, c) $\omega=0.3$ rad/s (not to scale).

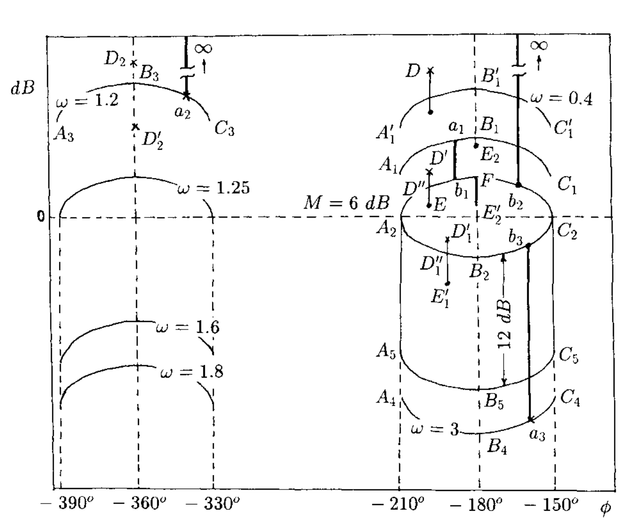

Fig. 4 Typical design boundaries (not to scale).

For frequencies above 2 rad/s, the templates consist of points along the vertical through -360 deg (for example, in the template corresponding to $\omega=3$ rad/s in Fig. 3, the nominal is marked x). The length of the templates for $\omega>2$ decreases from infinity as ω increases, until for large ω, $P(j\omega)\rightarrow k/\omega^4$, so the template reduces to 12 dB in length.

Boundary Generation for $|L_o(\omega)|$

As discussed previously, the objective is to assure that the entire family of transfer functions will have a minimum phase margin of 30 deg that is guaranteed by insuring that the closed-loop magnitude is less than 6 dB. In fact, by guaranteeing that the closed-loop magnitude of the entire family does not exceed 6 dB, a minimum phase margin of 30 deg and a minimum gain margin of about 4 dB are assured for the entire family. Shown in Fig. 4 is a gain-phase sketch showing some typical design boundaries for the nominal loop transfer function $L_o(j\omega)$. (Note that Fig. 4 is not drawn to scale.) To better understand this figure, consider three distinct frequency templates for $\omega=0.2$, 1.2, and 3.0 rad/s. The other boundaries corresponding to frequencies less than 1 rad/s, between 1 and 2 rad/s, and greater than 2 rad/s can be similarly obtained.

Boundary for $\omega=0.2$ rad/s

Since the requirement is that the loop transfer function should not enter the M-circle corresponding to 6 dB, the nominal loop transfer function $L_o(0.2j)$ should either lie above the curve $A_1B_1C_1$ or should lie completely below the curve $A_2B_2C_2$. Note that if a point D lying above $A_1B_1C_1$ is chosen as the nominal loop transfer function value at 0.2 rad/s, then all possible closed-loop magnitudes for varying k will be less than 6 dB since none of the points belonging to the template (the vertical line segment connecting the x at D to the • at the bottom) penetrates the 6-dB M-contour. However, if a point such as D' is chosen for $L_o(2j)$, then there are certain plants in the family that exhibit closed-loop magnitudes greater than 6 dB. For the specific choice shown in Fig. 4, part $D''E$ of the line segment $D'E$ represents loop transfer function values violating the 6-dB constraint. A similar argument can be given for a representative point D_1'. The line segment $D_1'D_1''$ represents loop transfer function values violating the 6-dB constraint. If the nominal loop transfer function is chosen so that it coincides with a_1, then the lowest point b_1 of the template falls right on the 6-dB contour, indicating that the loop transfer function gain cannot be made any lower at that phase angle unless it is made lower than the gain corresponding to the lower value on $A_2B_2C_2$ at the same phase angle. Another typical lower boundary for $\omega=0.4$ rad/s is $A_1'B_1'C_1'$.

Boundary for $\omega=3.0$ rad/s

The same argument just given for $\omega=0.2$ rad/s will apply here also. The nominal loop transfer function must stay below the boundary $A_4B_4C_4$ to satisfy the 6-dB constraint at $\omega=3$ rad/s. Note that, if a_3 is chosen as the nominal loop transfer function value at $\omega=3$, then the extreme point b_3 of the corresponding template falls on the 6-dB M-contour. As ω gets very large, the upper bound will move upward, eventually converging onto the curve $A_5B_5C_5$ that is 12 dB below $A_2B_2C_2$. In fact, $A_2FC_2C_5B_5A_5$ defines the U-contour for the system. Note that for frequencies close to but greater than 2 rad/s the lower boundary will be close to $-\infty$ dB.

Boundary for $\omega=1.2$ rad/s

Since the template consists of two vertical lines at -180 and -360 deg and the points at $+\infty$, the boundary consists of $A_3B_3C_3$ and the upper boundary of the 6-dB M-contour denoted $A_2D''FC_2$. Note that $L_o(1.2j)$ should lie above these two boundaries. There is no upper boundary as in the previous two cases because the template consists of points at $+\infty$. Consider the point D_2 as a choice for $L_o(1.2j)$. If we place the template for $\omega=1.2$ so that the nominal point, marked x, coincides with D_2, we notice that the bottom of the right leg E_2, marked •, of the template does not enter the 6-dB contour. But if we place the nominal point of the template at D_2', then we see a violation of the 6-dB M-contour by the right leg of the template. The line segment $E_2'F$ corresponds to members of the family of plants all violating the 6-dB condition. If the nominal loop transfer function is chosen to be at a_2 for $\omega=1.2$, then the lowest point b_2 of the right leg of the template falls on the 6-dB contour. As the frequency increases from 1 to 2, the allowed lower boundaries shift downward as indicated on Fig. 4.

Remark

There are no bounds on the nominal loop transfer function gain for -150 deg$\leq ArgL_o(j\omega)\leq 0$ deg and -330 deg $\leq ArgL_o(j\omega)\leq -210$ deg. This can be easily verified by noting that the templates, with the nominal point, marked x, placed at an assumed nominal loop gain in the defined region, never penetrate the 6-dB M-contour.

Loop Shaping $L_o(s)$

Once the design boundaries shown in Fig. 4 are obtained, a nominal loop transfer function must be shaped to satisfy the boundary requirements. In theory, infinitely many nominal loop transfer functions can be chosen to satisfy the gain-phase requirements on $L_o(s)$. However, some of them will have higher compensator bandwidths than the others.

The objective is to satisfy the constraints on the nominal loop gain (Fig. 4) but decrease $|L_o(j\omega)|$ as rapidly as possible.

One option is to have $|L_o|$ large enough in the low-frequency range to satisfy bounds by keeping -150 deg$< Arg L_o(j\omega) < 150$ deg until about $\omega = 10$ and only then allow $Arg L_o(j\omega) < -150$ deg and safely cross below the $A_5B_5C_5$ boundary to the origin. This would be realistic if such a relatively large bandwidth for L_o could be allowed and if P had no higher order highly underdamped unmodeled modes. Since the problem statement does not specify such unmodeled dynamics or specific limitations on the controller bandwidth, we will pursue the previous strategy for loop shaping, keeping in mind that to satisfy the settling time requirement stipulated in the specifications we need to insure a closed-loop bandwidth of about 0.9 rad/s. One such nominal loop transfer function is given as follows:

$$L_{o1}(s) = 0.4045 \times \frac{[1+(s/0.66)][1+(s/2.24)][1+(0.71s/0.813)+(s^2/0.813^2)]}{s^2(s^2+1)[1+(s/6.66)][1+(1.85s/5.877)+(s^2/5.877^2)][1+(0.75s/40)+(s^2/40^2)]} \tag{8}$$

The compensator transfer function can be extracted from $L_o(s)$ by dividing through by the nominal plant transfer function $0.5/[s^2(s^2+1)]$. The compensator has four zeros at -0.666, -2.24, and $-0.29 \pm j0.76$ and five poles at -6.66, $-5.45 \pm j2.2$, and $-15 \pm j37$.

Simulations

Figures 5–7 show computer simulations for a unit impulse disturbance in $w(t)$. Figure 5 shows the output $z(t)$ and $x_1(t)$ for the plant with parameter values $m_1 = m_2 = k = 1$ stipulated in the problem statement, and Fig. 6 shows the corresponding control input with a peak value less than 150. Several impulse responses for different values of k are shown in Fig. 7. From the simulation results, we can conclude that all of the quantitative design specifications (i.e., settling time of ≈ 15 s and robust stability) have been adequately met. We have also met the qualitative requirement of noise sensitivity by guaranteeing the controller bandwidth is finite. The final compensator is a fifth-order one, with four minimum phase zeros. A root locus plot vs the parameter $k \in [0.5, 2]$ is shown in Fig. 8.

QFT Design for Problem 2

To carry out a QFT design for problem 2, we first augment the plant transfer function with the internal model $1/(s^2+0.5^2)$ of the known sinusoidal disturbance. After that, we robustly stabilize the augmented plant. Since the uncertainty is only in the plant, we can use the same plant templates as the previous design, and because the settling time is 20 s, as a rule of thumb we will use the same minimum phase margin of 30 deg. Consequently, we obtain the same design boundaries as the first design. However, to account for the internal model, the nominal loop transfer function must now include the term $1/(s^2+0.5^2)$. A nominal loop transfer function that satisfies all of the constraints is given as follows:

$$L_{o2}(s) = \frac{0.125[1+(s/0.6)]^2[1+(s/0.5)][1+(s/10)][1+(s/0.5)+(s^2/0.5^2)]}{s^2(s^2+1)(s^2+0.5^2)[1+(s/5.0)][1+(s/50)+(s^2/50^2)]^2[1+(0.8s/50)+(s^2/50^2)]} \tag{9}$$

Remark

The same strategy as before (for L_{o1}) was used for loop shaping $L_{o2}(s)$. The compensator can be extracted from the previous nominal loop transfer function by dividing through by the nominal plant transfer function $0.5/[s^2(s^2+1)]$. The compensator has six zeroes at -0.5, -0.6, -0.6, -10.0, and $-0.25 \pm 0.43j$ and nine poles at $\pm 0.5j$, -5, $-20 \pm 45.8j$, $-25 \pm 43.3j$, and $-25 \pm 43.3j$.

Simulation

Figures 9–12 show simulation results with the compensator in the loop. Figure 9 shows the output $z(t)$ and $x_1(t)$ resulting from a sinusoidal disturbance at frequency 0.5 rad/s for the plant with $m_1 = m_2 = k = 1$. The resulting control effort is given in Fig. 10 with the peak less than 3. Figure 11 shows several output responses due to the sinusoidal input for varying values of k. Also shown is the output due to an impulse for $k = 0.5$ and 1.0 in Fig. 12. The compensator for the second design problem is ninth order with six zeros. A root locus plot vs the parameter $k \in [0.5, 2]$ for design 2 is given in Fig. 13.

Problem 3

The purpose of this design is to maximize a stability performance measure with respect to the three uncertain parameters m_1, m_2, and k. The nominal values are assumed to be $m_1 = m_2 = 1$ and $k = 0.5$, the same as design 1. We will restrict the solution to that of determining allowable percent variations in the three parameters with the controller of design 1 in the loop. We will give a quick estimate only. Better estimates are possible but are tedious to obtain (since QFT is a technique best suited for design synthesis and not for analysis).

Estimating Allowable Parameter Variations

In developing the boundaries for design 1, we allowed the plant to have independent parameter variations in the numerator and the denominator of the form

$$P(s) = \frac{k}{s^2(s^2+2k_1)}, \quad 0.5 < k < 2, \quad \text{and} \quad 0.5 < k_1 < 2 \tag{10}$$

although we could have used the fact that $k = k_1$. This was deliberately avoided to show how a quick estimate of the allowable parameter variations can be obtained. To get the necessary estimate, we rewrite the plant transfer function in the following form:

$$P(s) = \frac{(k/m_1m_2)}{s^2\{s^2+[(m_1+m_2)k/m_1m_2]\}} \tag{11}$$

Now, by comparing Eqs. (10) and (11), we can write down conditions for allowable parameter variations as

$$0.5 < \frac{k}{m_1m_2} < 2 \tag{12}$$

and

$$1 < \left(\frac{m_1+m_2}{m_1m_2}\right)k < 4 \tag{13}$$

On examining the first of the last two conditions, we see that only allowable variations are those in which k is allowed to increase and m_1 and m_2 are allowed to decrease. This is so because we had chosen $k = 0.5$, the smallest value for k, as the nominal. If we assume the same percent $\delta\%$ increase in k and decrease in each m_1 and m_2, then clearly the lower bound on the parameter grouping is always satisfied. To determine the largest allowable δ, write

$$\frac{0.5(1+\delta)}{(1-\delta)^2} < 2$$

leading to

$$4\delta^2 - 9\delta + 3 > 0$$

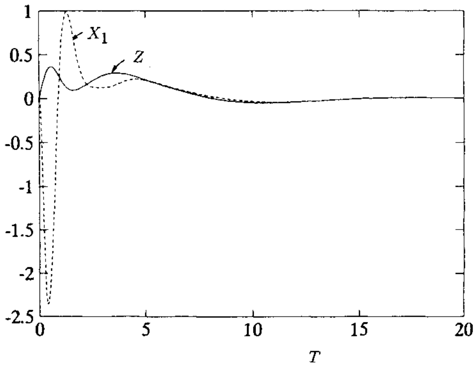

Fig. 5 x_1 vs t and z vs t for a unit impulse in w for the plant $m_1 = m_2 = 1$ and $k = 1$.

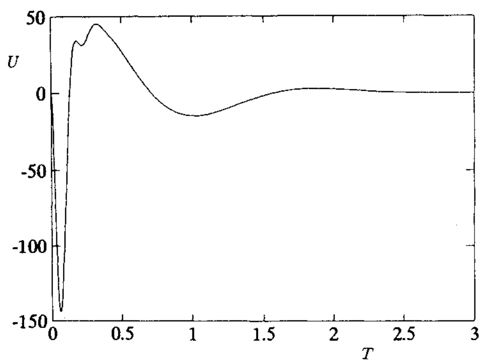

Fig. 6 Control input u vs t for the transient response of Fig. 5.

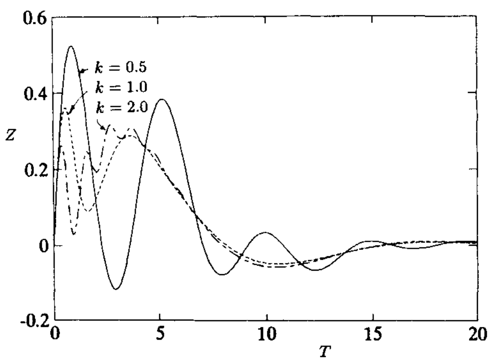

Fig. 7 z vs t for a unit impulse in w with $m_1 = m_2 = 1$ and $k = 0.5$, 1.0, and 2.0.

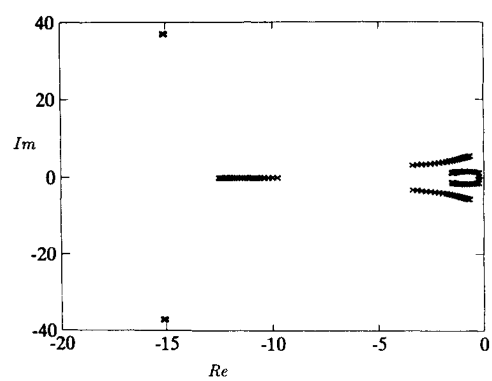

Fig. 8 Root locus of $L_{o1}(s)$ vs $k \in [0.5, 2]$.

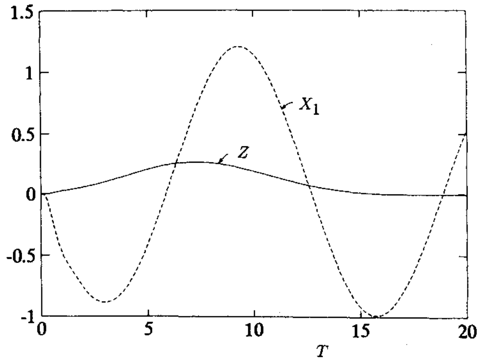

Fig. 9 z vs t and x_1 vs t for the sinusoidal disturbance $w(t) = \sin 0.5t$ for the plant $m_1 = m_2 = 1$ and $k = 1$.

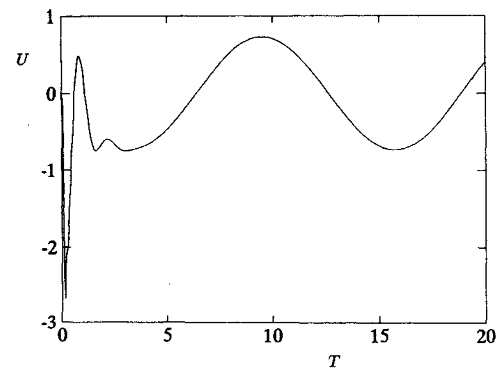

Fig. 10 Control input u vs t for the transient response of Fig. 9.

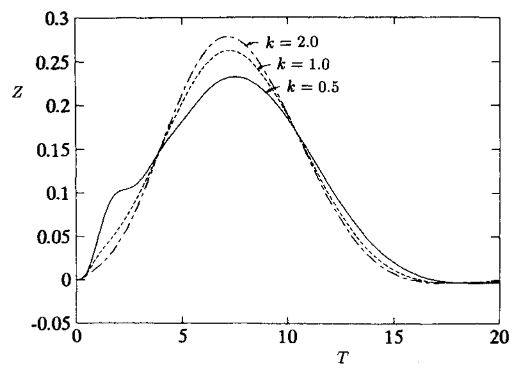

Fig. 11 z vs t for $w(t) = \sin 0.5t$ with $m_1 = m_2 = 1$ and $k = 0.5$, 1.0, and 2.0.

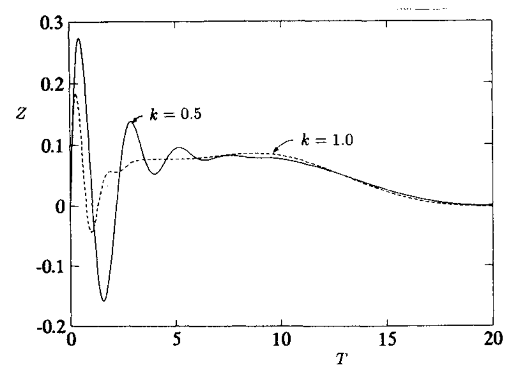

Fig. 12 z vs t for a unit impulse in w with $m_1 = m_2 = 1$ and $k = 0.5$, 1.0.

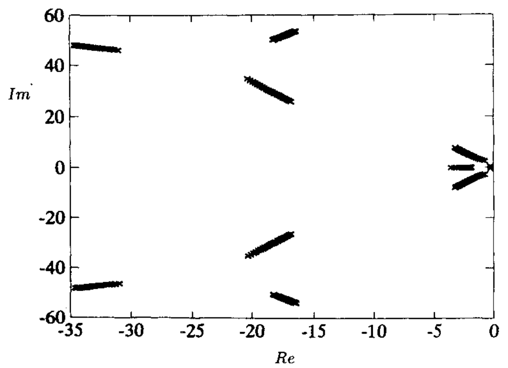

Fig. 13 Root locus of $L_{o2}(s)$ vs $k \in [0.5, 2]$.

implying that $\delta < 0.40$ or $\delta > 1.8$. Next we will examine the second condition of Eq. (13) and write it in the form

$$1 < \frac{(2-\delta)}{(1-\delta)^2} 0.5(1+\delta) < 4$$

leading to

$$9\delta^2 - 17\delta + 6 > 0$$

$$\delta(\delta - 1.66) < 0$$

from which it follows that $\delta < 0.47$. Hence we conclude that the allowable perturbations in parameters is limited to 40%. In particular, the following parameter variations can be allowed while satisfying the performance requirements stipulated in design 1:

$$0.5 < k < 0.7, \quad 0.6 < m_1 < 1, \quad \text{and} \quad 0.6 < m_2 < 1$$

Equations (12) and (13) give a large class of possible parameter variations, including the ones given earlier.

These estimates will satisfy all of the performance constraints and are in fact conservative estimates of the possible parameter variations when the controller for design 1 is in the loop.

IV. Discussion

Presented in this paper are QFT designs for the benchmark problem. The two QFT designs satisfy the quantitatively given design specifications of settling time and robust stability as was shown in computer simulations. Although a special attempt was not made to minimize the bandwidth of the synthesized controllers, they certainly have finite bandwidths. If controller bandwidth is not an issue (i.e., one has perfect sensors), then the controller order can be reduced. For example, without much trouble both compensators can be made differentiators by neglecting the high-frequency poles of L_{o1} and L_{o2}. But if that is done, the controller bandwidth goes to ∞ and the system will be highly sensitive to noise. One of the powerful attributes of QFT is that the controller complexity can be quite easily related to the cost of feedback as measured by controller bandwidth. It is generally true that the controller complexity will increase inversely with the controller bandwidth, a key consideration in any practical design and QFT in particular.

An estimate of the tolerable parameter variations was given under design 3. It is generally not easy to estimate the extent of tolerable parameter uncertainty for a given controller using the QFT approach. This is especially true if the number of uncertain parameters is large. Note that, in a QFT design, enlarging the range of parameter uncertainty is equivalent to changing the shape and size of the templates. If we agree to change all parameters by a certain percentage $\pm\delta$, then we can iterate on δ until the nominal loop transfer function barely touches one of the allowable design boundaries at some frequency. This can be done systematically as follows. At each frequency ω determine the maximum value $\delta_{\max}(\omega)$ of $\delta(\omega)$ so that the nominal loop transfer function barely enters a forbidden region. Repeat the procedure at all relevant frequencies to determine the maximum tolerable δ given by the minimum over all of the $\delta_{\max}(\omega)$, $\omega \in [0, \omega_c]$, where ω_c defines the useful frequency range.

One revealing feature of the estimates we made earlier in design 3 is the fact that the stiffness of the spring cannot be allowed to decrease further because we had designed our controllers based on the worst plant in the set (in the sense of having the lowest undamped natural frequency). It should be emphasized that our QFT designs are a bit conservative with respect to the performance specifications stipulated for the specific plant ($m_1 = m_2 = k = 1$) because we chose the softest plant ($m_1 = m_2 = 1$ and $k = 0.5$) as our nominal.

V. Conclusion

Problems 1 and 2 of the benchmark problem were solved using the technique of quantitative feedback theory. Both designs adequately met the settling time and robust stability requirements. Both controllers designed have finite bandwidth, thus insuring insensitivity to noise beyond a certain frequency. Although no special attempt was made to minimize the control effort or the controller bandwidth, they both appear to be reasonable. Design 3 was not attempted because quantitative feedback theory is not well suited for such problem statements. Instead, the range of uncertainties that can be tolerated with the controller of design 1 was estimated.

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